Analytic heliospheric magnetic field modeling

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Talk outline:

1. The Base Model
   - Introduction and Idea
   - Solution Properties
2. Improvements
   - Extension I: Non-circular Cross Sections
   - Extension II: Compressible Flow
   - (Some ideas for) Extension III: The Inside

Goal: Find an analytical model for the interstellar magnetic field draping around the heliopause (HP). Useful for many purposes. Only very few such models exist:

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Ansatz: $\mathbf{B}$ should satisfy $\nabla \cdot \mathbf{B} = 0$ as well as $\nabla \times (\mathbf{u} \times \mathbf{B}) = \mathbf{0}$, with boundary condition $\mathbf{B}|_{z\to\infty} = \mathbf{B}_0$.

Prescribed flow: "*Rankine half-body*"
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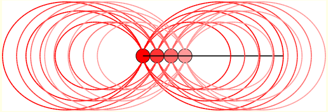

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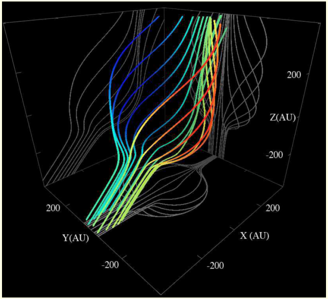

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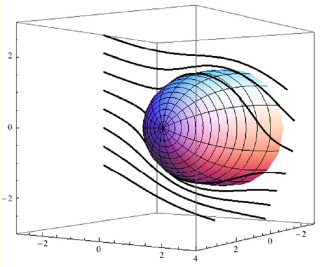

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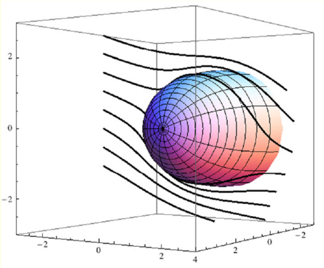

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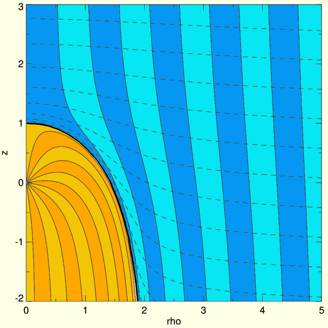
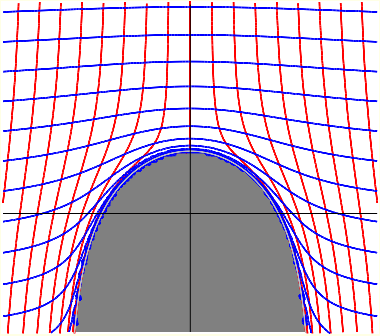


Key idea

- Stream lines and isochrones (lines of constant travel time T) form a non-orthog. coordinate system $\mathcal{K}$ exterior to HP.
- Advected **B** field components are constant w. r. t. $\mathcal{K}$ due to line conservation [Cauchy 1816].



Main difficulty

Analytic evaluation of ...along stream line

$$T(r,\vartheta) = \int_\infty^r \frac{\mathrm{d}r'}{u_r(r')} = \int_0^\vartheta \frac{r(\vartheta')\,\mathrm{d}\vartheta'}{u_\vartheta(\vartheta')}$$

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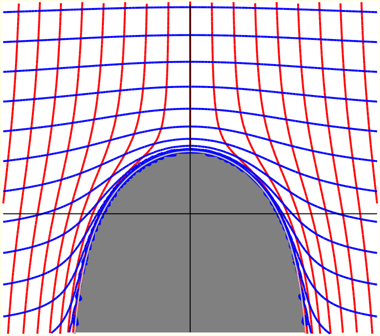

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The exact solution [Röken+, ApJ 2015]

$$B_\rho(\rho,\varphi,z) = -\frac{q\rho}{r^3}\,B_{z0} + \left[\frac{q^{3/2}\rho}{r^3 a}\,\mathcal{T} + \frac{a}{\rho}\left(1+\frac{qz}{r^3}\right)\right] B_{\rho 0}$$

$$B_\varphi(\rho,\varphi,z) = \frac{\rho}{a}\,B_{\varphi 0}$$

$$B_z(\rho,\varphi,z) = \left(1-\frac{qz}{r^3}\right) B_{z0} + \left[\left(\frac{qz}{r^3}-1\right)\frac{\sqrt{q}}{a}\,\mathcal{T} + \frac{qz^2 a}{r^3\rho^2}\right] B_{\rho 0}$$

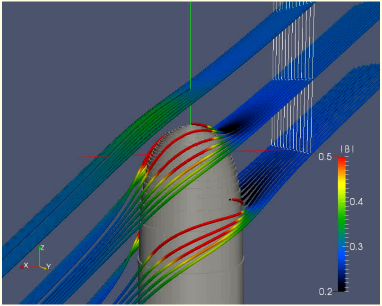
with $\mathbf{B}_0 = \mathbf{B}|_\infty$, incomplete elliptic integrals $\{F, E\}$, and

$$\mathcal{T} := \left(2-\kappa^{-2}\right) E(\lambda,\kappa) - \left(1-\kappa^{-2}\right) F(\lambda,\kappa) \quad\Big|\quad \lambda := \sqrt{1-(a/\rho)^2}$$

$$a := \sqrt{\rho^2 + 2q\,(z/r-1)} \quad\Big|\quad \kappa := \sqrt{1+a^2/(4q)}$$

Visualization of field line structure shows expected behavior:

- undisturbed ISM field at large distances from HP
- field lines do not penetrate HP, but drape around it, eventually becoming tangential to HP
- draping increases field strength (reaching ∞ on HP!)



Model extension I: Non-circular cross sections

- **u** axially symmetric $\Rightarrow$ heliotail has circular cross section (despite 'squeezing' due to $\mathbf{B}_{\rm ism}$). This is somewhat unrealistic...
- ...but can be fixed (i.e., adjusted to tailor-made aspect ratio $\eta(z) := a/b$ and area ($\propto a\, b$) using a distortion flow **w** ($\neq$ **u**!) and still remain exact if $\nabla(\nabla \cdot \mathbf{w}) = \mathbf{0}$. [Kleimann+, ApJ 2016]

w field on z planes original **B** distorted **B**

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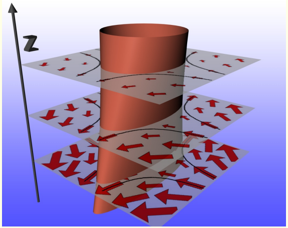

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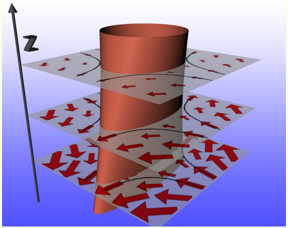
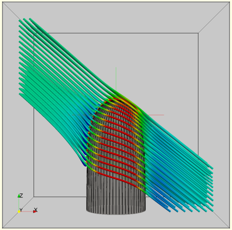


w field on *z* planes

⇒



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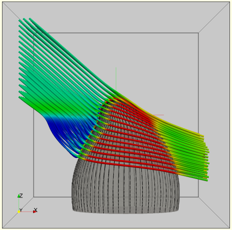

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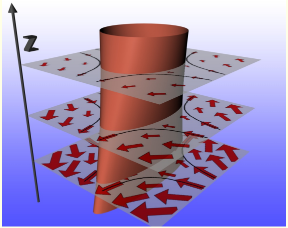

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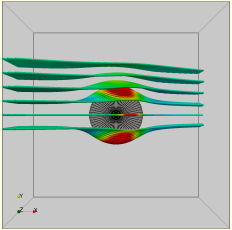

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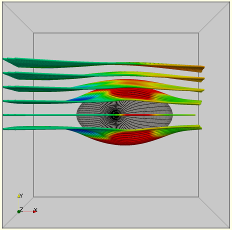

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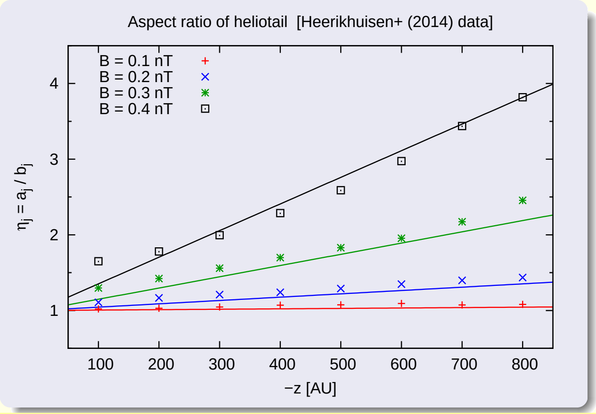
Realistic parameters for $\eta(z, B_{\rm ism})$ from simulations



$$\eta_{\rm fit}(z, B_{\rm ism}) = 1 + 3.27 \left(\frac{-z}{100\ \text{AU}}\right) \left(\frac{B_{\rm ism}}{\text{nT}}\right)^{2.5}$$

Model extension II: Compressible flow

Issues:
1. Incompressible flow condition questionable.
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Requires: System closure through Bernoulli's law
$(\mathbf{u} \cdot \nabla)\mathbf{u} = -(\nabla P)/n$ along flow lines,
plus equation of state $P \propto n^\gamma$, $\gamma \in \{1, 5/3\}$

Key findings (density / flow field)

1. New parameter: upstream Mach number $m \in [0, 1[$.
   ($m = 0 \Leftrightarrow n = \text{const.}$)
2. n^2 can be expressed using Lambert's W function ($\gamma = 1$), or as root of a fourth-order polynomial ($\gamma = 5/3$).
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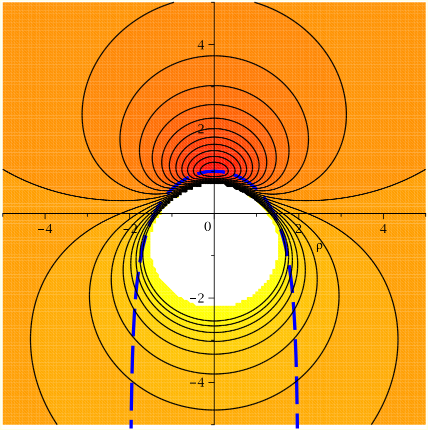

n contours ($\gamma = 1$, $m = 0.6$)

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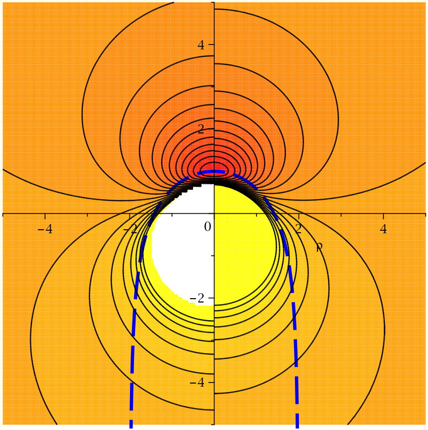
3. Density looks plausible; reasonable polynomial approximation for n



n_{exact} vs. n_{approx} ($\gamma = 1$, $m = 0.6$)

Key findings (magnetic field) [Kleimann+, ApJ 2017]

1. 'Compressible' ($m > 0$) **B** solution (w.r.t. $\mathbf{u}_{\mathrm{approx}}$) can be expressed through known **B** solution of the $m = 0$ case (without using any "exotic" functions).
2. **B** now stronger in pile-up region near stagnation point.
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NB: Approximation merely concerns momentum conservation. **B** thus derived from $\mathbf{u}_{\mathrm{approx}}$ is again an exact MHD solution!

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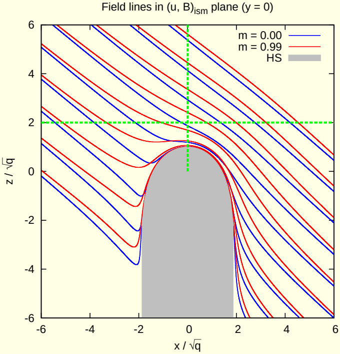
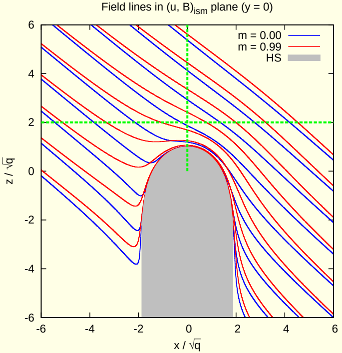
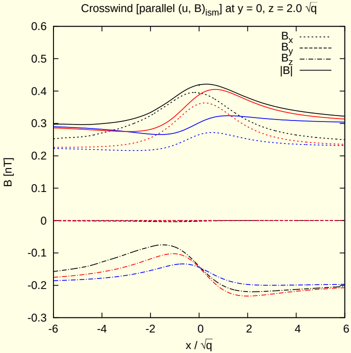
Comparison $\mathbf{B}_{\rm old} \leftrightarrow \mathbf{B}_{\rm new}$

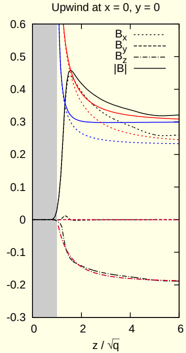


Field lines (compress. vs. incompressible)

Comparison $\mathbf{B}_{\mathrm{old}} \leftrightarrow \mathbf{B}_{\mathrm{new}} \leftrightarrow$ self-consistent MHD sim.







Field lines (compress. vs. incompressible)

Cut along x and z (numerics in black)

Note blue → red → black ordering!

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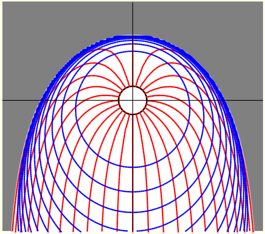
Goal: Extend method to the actual heliosphere (= HP interior). In principle, same PDEs with different boundary conditions.



Difficulties

1. stream lines kink at termination shock (TS)
2. Inside TS: diverging $\|\nabla\Phi\| \propto 1/r^2$
3. realistic time-dependent solar cycle BCs that leave $\nabla \cdot \mathbf{B} = 0$ intact

Minimal working example: magnetic rings

Parker spiral has $B_\varphi \gg B_r$ almost everywhere.

$\mathbf{B}_0(\mathbf{r}_0, t) = B_0(r, \vartheta', t)\, \mathbf{e}_{\varphi'}$ in Sun system K′

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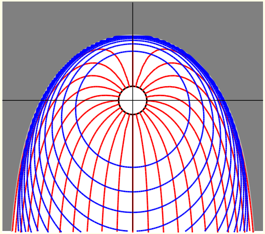

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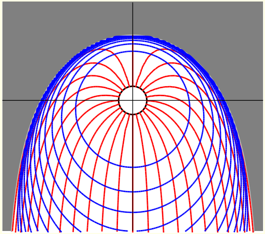

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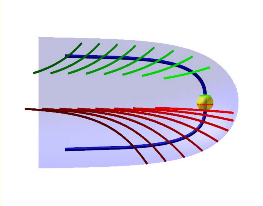

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Conclusions

- Existing exact analytical (2015) model of the local interstellar magnetic field frozen into the Rankine flow.
- Two-fold extension to (i) non-circular heliotail cross sections and (ii) mildly compressible flow without loss of generality or analytical tractability
- Exact(!) compressible magnetic field solution to now even closer to self-consistent MHD numerics.
- Extension to interior heliosphere ongoing.
  Tricky because of – termination shock treatment
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- But: Good prospects for simpler "smoke ring" approach!

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BACKUP SLIDES

A note on the approximation of non-existent solutions

Extension into "white" region

- − lacks a measure of departure from "real" flow solution, but
- + follows stream lines,
- + conserves mass,
- + and looks as expected.

Local(!) Mach number for $m \in \{0.25, 0.5, 0.75, 1.0\}$ along HP (Marks for up/cross/downwind)

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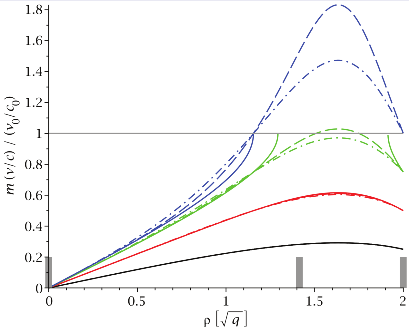
NB: $\|\mathbf{u}_{\text{appr.}}\|/c_s \leq 1.83$
(supersonic, despite $m < 1$)



Local(!) Mach number for $m \in \{0.25, 0.5, 0.75, 1.0\}$ along HP (Marks for up/cross/downwind)